

ON UNIVERSAL ENVELOPING ALGEBRAS IN A TOPOLOGICAL SETTING

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ABSTRACT. We establish the exponential law for suitably topologies on spaces of vector-valued smooth functions on topological groups, where smoothness is defined by using differentiability along continuous one-parameter subgroups. As an application, we investigate the canonical correspondences between the universal enveloping algebra, the invariant local operators, and the convolution algebra of distributions supported at the unit element of any finite-dimensional Lie group, when one passes from finite-dimensional Lie groups to pre-Lie groups. The latter class includes for instance any locally compact groups, Banach-Lie groups, additive groups underlying locally convex vector spaces, and also mapping groups consisting of rapidly decreasing Lie group-valued functions.

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1. INTRODUCTION

It is well-known that Lie theory and the related representation theory have been successfully extended much beyond the classical setting of finite-dimensional real Lie groups, and this research area now includes locally compact groups ([HM07], [HM13]), Lie groups modeled on Banach spaces or even on locally convex spaces ([KM97], [Bel06], [Ne06]), and some other classes of topological groups which may not be locally compact ([BCR81], [Glö02b], [HM05]). The differential calculus on topological groups, involving functions which are smooth along the one-parameter subgroups (Definition 2.3), plays an important role for these extensions of Lie theory and has recently found remarkable applications also to supergroups and their representation theory ([NS13a], [NS13b]). We have merely mentioned here a very few references that are closer related to the topics of our paper.

On the other hand, as one can see for instance in [Wa72] or [Ped94], a key fact in harmonic analysis and representation theory is that the universal enveloping algebras

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of finite-dimensional Lie algebras can be realized by linear functionals or operators on spaces of smooth functions on the corresponding Lie groups, for instance as convolution algebras of distributions supported at the unit element or as invariant linear differential operators. It is then natural to seek for such realizations beyond the classical setting of finite dimensional Lie groups, with motivation coming from the representation theory of groups of the aforementioned types. In the present paper we begin an investigation on that question, oriented towards a pretty large class of topological groups which have sufficiently many one-parameter subgroups, namely the pre-Lie groups; see Definition 5.4 and Examples 5.6–5.9 below. (A sequel paper will deal with the situation when the domain \mathbb{R} of the one-parameter groups is replaced by suitable subsets of more general topological fields, to some extent in the spirit of [BGN04], [BN05], and [Ber08].)

To this end, one needs a suitable notion of distributions with compact support, that is, continuous linear functionals on the space of smooth functions of the group under consideration. While spaces of smooth functions on any topological group were already studied in the earlier literature, one still needs to give these function spaces a topology adequate for the purposes of turning their topological duals into associative algebras which act on function spaces by the natural operation of convolution. It should be pointed out here that although the convolution of functions on a topological group requires some Haar measure on that group, this is not necessary for the convolution of functions with distributions or measures (see Definition 3.5).

One of the main technical novelties of our paper is the construction of a suitable topology on the space of smooth functions on any topological group and with values in any locally convex space \mathcal{Y} , for which for arbitrary topological groups G and H the exponential law for smooth functions

$$\mathcal{C}^\infty(H \times G, \mathcal{Y}) \simeq \mathcal{C}^\infty(H, \mathcal{C}^\infty(G, \mathcal{Y}))$$

holds true (see Theorem 4.16 and Remark 4.17 below). By using that fact, we then prove that for any pre-Lie group G , the convolution with distributions with compact support (that is, linear functionals which are continuous for the aforementioned suitable topology) does preserve the space of smooth functions $\mathcal{C}^\infty(G)$ (Proposition 5.1). By focusing on distributions supported at $\mathbf{1} \in G$, we can thus identify them with continuous linear operators on $\mathcal{C}^\infty(G)$ which commute with the left translations and are local, in the sense that they do not increase the support of functions (Theorem 5.2). Recall that Peetre's theorem from [Pee60] ensures that the local operators on smooth manifolds are precisely the differential operators, not necessarily of finite order. If G is any finite-dimensional Lie group, then we recover the natural correspondence between the distributions supported at $\mathbf{1} \in G$ and the left invariant differential operators on G .

The topology that we introduce on any function space $\mathcal{C}^\infty(G, \mathcal{Y})$ agrees with the topology of uniform convergence of functions and their derivatives if G is any finite-dimensional real Lie group. However, unlike the most constructions of similar topologies on spaces of test functions from the literature, our construction (Definition 3.1) does not need the group G to be locally compact. In fact, spaces of test functions, distributions, and universal enveloping algebras were already investigated on locally compact groups which were not necessarily Lie groups, for instance:

- Basic distribution theory on abelian locally compact groups by using differentiability along one-parameter subgroups was developed in [Ri53].
- Let G be any topological group which is a projective limit of Lie groups. Under the additional hypotheses that G is simply connected, locally compact, and separable, one endowed the space $\mathcal{C}^\infty(G)$ in [Ka59], [Ka61], [Ma61], [Br61], [BC75, Sect. 2] with the topology of a locally convex space, which is nuclear if and only if every

quotient group of G whose Lie algebra is finite-dimensional is necessarily a Lie group, as proved in [BC75, Sätze 3.3, 3.5].

- Some nuclear function spaces on locally compact groups that do not use approximations by Lie groups were constructed in [Py74].
- Universal enveloping algebras of separable locally compact groups which are projective limits of Lie groups were studied in [Br61], [MM64], and [MM65].
- More recently, differential operators and their relation to distributions and convolutions on locally compact groups were also studied in [Ed88] and [Ak95].

Our article is organized as follows: In Section 2 we provide some basic definitions and auxiliary results from the differential calculus on topological groups. Section 3 introduces the convolution of smooth functions with compactly supported distributions and states one of the main problems which motivated the present investigation (Problem 3.14). Section 4 is devoted to proving the exponential law for smooth functions on topological groups (Theorem 4.16), which is our main technical result. Finally, in Section 5 we use that technical result for establishing the structure of invariant local operators (Theorem 5.2).

General notation. Throughout the present paper we denote by G , H arbitrary topological groups, unless otherwise mentioned. We will assume that the topology of any topological group is separated. For any topological spaces T and S we denote by $\mathcal{C}(T, S)$ the set of all continuous maps from T into S .

2. PRELIMINARIES

This section presents some ideas and notions of Lie theory that play a key role in the present paper. Our basic references for Lie theory of topological groups are [BCR81], [HM05], and [HM07].

The adjoint action of a topological group. Let G be any topological group with the set of neighborhoods of $\mathbf{1} \in G$ denoted by $\mathcal{V}_G(\mathbf{1})$. Define

$$\mathfrak{L}(G) = \{\gamma \in \mathcal{C}(\mathbb{R}, G) \mid (\forall t, s \in \mathbb{R}) \quad \gamma(t+s) = \gamma(t)\gamma(s)\}.$$

We endow $\mathfrak{L}(G)$ with the topology of uniform convergence on the compact subsets of \mathbb{R} . It can be described by neighborhood bases as follows. For arbitrary $n \in \mathbb{N}$ and $U \in \mathcal{V}_G(\mathbf{1})$ denote

$$W_{n,U} = \{(\gamma_1, \gamma_2) \in \mathfrak{L}(G) \times \mathfrak{L}(G) \mid (\forall t \in [-n, n]) \quad \gamma_2(t)\gamma_1(t)^{-1} \in U\}.$$

For every $\gamma_1 \in \mathfrak{L}(G)$ define $W_{n,U}(\gamma_1) = \{\gamma_2 \in \mathfrak{L}(G) \mid (\gamma_1, \gamma_2) \in W_{n,U}\}$. Then there exists a unique topology on $\mathfrak{L}(G)$ with the property that for each $\gamma \in \mathfrak{L}(G)$ the family $\{W_{n,U}(\gamma) \mid n \in \mathbb{N}, U \in \mathcal{V}_G(\mathbf{1})\}$ is a fundamental system of neighborhoods of γ .

Definition 2.1. The *adjoint action* of the topological group G is the mapping

$$\text{Ad}_G: G \times \mathfrak{L}(G) \rightarrow \mathfrak{L}(G), \quad (g, \gamma) \mapsto \text{Ad}_G(g)\gamma := g\gamma(\cdot)g^{-1}.$$

Since the action of G on itself by inner automorphisms $G \times G \rightarrow G$, $(g, h) \mapsto ghg^{-1}$, is continuous, it follows that the above mapping Ad_G indeed takes values in $\mathfrak{L}(G)$ and is a group action.

We now recall the following result for later use:

Lemma 2.2. *The adjoint action of every topological group is a continuous mapping.*

Proof. See [BCR81, Lemma 0.1.4.1]. □

Differentiability along one-parameter subgroups.

Definition 2.3. Let G be any topological group with an arbitrary open subset $V \subseteq G$ and \mathcal{Y} be any real locally convex space. If $\varphi: V \rightarrow \mathcal{Y}$, $\gamma \in \mathfrak{L}(G)$, and $g \in V$, then we denote

$$(D_\gamma^\lambda \varphi)(g) = \lim_{t \rightarrow 0} \frac{\varphi(g\gamma(t)) - \varphi(g)}{t} \quad (2.1)$$

if the limit in the right-hand side exists.

We define $\mathcal{C}^1(V, \mathcal{Y})$ as the set of all $\varphi \in \mathcal{C}(V, \mathcal{Y})$ for which the function

$$D^\lambda \varphi: V \times \mathfrak{L}(G) \rightarrow \mathcal{Y}, \quad (D^\lambda \varphi)(g; \gamma) := (D_\gamma^\lambda \varphi)(g)$$

is well defined and continuous. We also denote $D^\lambda \varphi = (D^\lambda)^1 \varphi$.

Now let $n \geq 2$ and assume the space $\mathcal{C}^{n-1}(V, \mathcal{Y})$ and the mapping $(D^\lambda)^{n-1}$ have been defined. Then we define $\mathcal{C}^n(V, \mathcal{Y})$ as the set of all functions $\varphi \in \mathcal{C}^{n-1}(V, \mathcal{Y})$ for which the function

$$(D^\lambda)^n \varphi: V \times \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \rightarrow \mathcal{Y}, \\ (g; \gamma_1, \dots, \gamma_n) \mapsto (D_{\gamma_n}^\lambda (D_{\gamma_{n-1}}^\lambda \cdots (D_{\gamma_1}^\lambda \varphi) \cdots))(g)$$

is well defined and continuous.

Moreover we define $\mathcal{C}^\infty(V, \mathcal{Y}) := \bigcap_{n \geq 1} \mathcal{C}^n(V, \mathcal{Y})$. If $\mathcal{Y} = \mathbb{C}$, then we write simply $\mathcal{C}^n(G) := \mathcal{C}^n(V, \mathbb{C})$ etc., for $n = 1, 2, \dots, \infty$.

Notation 2.4. It will be convenient to use the notation

$$D_\gamma^\lambda \varphi := D_{\gamma_n}^\lambda (D_{\gamma_{n-1}}^\lambda \cdots (D_{\gamma_1}^\lambda \varphi) \cdots): G \rightarrow \mathcal{Y}$$

whenever $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G)$ and $\varphi \in \mathcal{C}^n(G, \mathcal{Y})$.

Some auxiliary facts. For later use we record the following well-known facts.

Lemma 2.5. *Let X and T be any topological spaces, \mathcal{Y} be any locally convex space, and $f: X \times T \rightarrow \mathcal{Y}$ be any continuous function. Pick any point $x_0 \in X$ and compact set $K \subseteq T$. Then for any continuous seminorm $|\cdot|$ on \mathcal{Y} we have*

$$\lim_{x \rightarrow x_0} \sup_{t \in K} |f(x, t) - f(x_0, t)| = 0. \quad (2.2)$$

Proof. This result is well known and is related to the exponential law for continuous functions $\mathcal{C}(X \times T, \mathcal{Y}) \simeq \mathcal{C}(X, \mathcal{C}(T, \mathcal{Y}))$; see for instance [AD51, Th. 4.21]. \square

In the following lemma we record the continuity with respect to parameters for the weak integrals in locally convex spaces which may not be complete; see [Gl02a] for a thorough discussion of that integral, related differential calculus, and their applications to Lie theory.

Lemma 2.6. *Let X be any topological space, \mathcal{Y} be any locally convex space, $a, b \in \mathbb{R}$, $a < b$, and $f: X \times [a, b] \rightarrow \mathcal{Y}$ be any continuous function with the property that for every $x \in X$ there exists the weak integral $h(x) = \int_a^b f(x, t) dt$. Then the function $h: X \rightarrow \mathcal{Y}$ obtained in this way is continuous.*

Proof. To prove that the function h is continuous, let $|\cdot|$ be any continuous seminorm on \mathcal{Y} . It follows by [Gl02a, Lemma 1.7] that we have

$$(\forall x, y \in X) \quad |h(x) - h(y)| \leq (b - a) \sup_{t \in [a, b]} |f(x, t) - f(y, t)|$$

and now by using Lemma 2.5 we readily see that the function $h: X \rightarrow \mathcal{Y}$ is continuous. \square

Lemma 2.7. *Let H be any topological group and $h \in \mathcal{C}(H, \mathcal{Y})$. If $X \in \mathfrak{L}(H)$ and the derivative $D_X^\lambda h: H \rightarrow \mathcal{Y}$ exists and is continuous, then there exists a continuous function $\chi: \mathbb{R} \times H \rightarrow \mathcal{Y}$ satisfying for arbitrary $g \in H$ the conditions*

$$(\forall t \in \mathbb{R}) \quad h(gX(t)) = h(g) + t(D_X^\lambda h)(g) + t\chi(t, g)$$

and $\chi(0, g) = 0$.

Proof. This follows by [NS13a, Lemma 2.5]; see also [BB11, Prop. 2.3]. \square

3. DISTRIBUTIONS WITH COMPACT SUPPORT, CONVOLUTIONS, AND LOCAL OPERATORS

In this section we give a precise statement of the problem that motivated the present paper; see Problem 3.14 below.

Topologies on spaces of smooth functions. Spaces of smooth functions and their topologies play an important role in the theory of infinite-dimensional Lie groups modeled on locally convex spaces; see for instance [Ne06, Def. I.5.1]. We will now introduce a suitable topology on spaces of smooth functions on any topological group G , by using compact subsets of the space of one-parameter subgroups $\mathfrak{L}(G)$ and its Cartesian powers. This topology turns out to be adequate for establishing the exponential law (Theorem 4.16 and Remark 4.17).

Definition 3.1. Let G be any topological group and denote

$$(\forall k \geq 1) \quad \mathfrak{L}^k(G) := \underbrace{\mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G)}_{k \text{ times}}.$$

Pick any open set $V \subseteq G$. If \mathcal{Y} is any locally convex space, then for every $k \geq 1$, any compact subsets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq V$, and any continuous seminorm $|\cdot|$ on \mathcal{Y} we define

$$p_{K_1, K_2}^{|\cdot|}: \mathcal{C}^\infty(V, \mathcal{Y}) \rightarrow [0, \infty), \quad p_{K_1, K_2}^{|\cdot|}(f) = \sup\{|(D_\gamma^\lambda f)(x)| \mid \gamma \in K_1, x \in K_2\}.$$

For the sake of simplicity we will always omit the seminorm $|\cdot|$ on \mathcal{Y} from the above notation, by writing simply p_{K_1, K_2} instead of $p_{K_1, K_2}^{|\cdot|}$.

We endow the function space $\mathcal{C}^\infty(V, \mathcal{Y})$ with the locally convex topology defined by the family of these seminorms p_{K_1, K_2} and the locally convex space obtained in this way will be denoted by $\mathcal{E}(V, \mathcal{Y})$. If $\mathcal{Y} = \mathbb{C}$ then we write simply $\mathcal{E}(V) := \mathcal{E}(V, \mathcal{Y})$.

We also denote by $\mathcal{E}'(G)$ the topological dual of $\mathcal{E}(G)$ endowed with the weak dual topology. This means that we have

$$\mathcal{E}'(G) = \{u: \mathcal{E}(G) \rightarrow \mathbb{C} \mid u \text{ is linear and continuous}\}$$

as a linear space, and this space of linear functionals is endowed with the locally convex topology defined by the family of seminorms $\{q_B \mid B \text{ finite} \subseteq \mathcal{E}(G)\}$, where for every finite subset $B \subseteq \mathcal{E}(G)$ we define the seminorm

$$q_B: \mathcal{E}'(G) \rightarrow \mathbb{C}, \quad q_B(u) := \max_{f \in B} |u(f)|.$$

The elements of $\mathcal{E}'(G)$ will be called *distributions with compact support* on G .

Before to go further, we state an interesting problem related to the above definition.

Problem 3.2. *Find conditions on the topological group G ensuring that every closed bounded subset of the locally convex space $\mathcal{E}(G)$ is compact.*

The above problem will not be addressed in the present paper. Let us just mention that it has an affirmative answer if G is any finite-dimensional Lie group; see [Eh56].

Definition 3.3. Assume the setting of Definition 3.1. The *support* of any $u \in \mathcal{E}'(G)$ is denoted by $\text{supp } u$ and is defined as the set of all points $x \in G$ with the property that for every neighborhood U of x there exists $f \in \mathcal{E}(G)$ such that $\text{supp } f \subseteq U$ and $u(f) \neq 0$.

Remark 3.4. For every $u \in \mathcal{E}'(G)$, by using its continuity with respect to the topology of $\mathcal{E}(G)$ introduced in Definition 3.1, it follows that there exist a positive constant $C > 0$, an integer $k \geq 1$, and some compact subsets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq G$ for which

$$(\forall f \in \mathcal{E}(G)) \quad |u(f)| \leq Cp_{K_1, K_2}(f).$$

This implies $\text{supp } u \subseteq K_2$, hence the set $\text{supp } u$ is compact in G , and this motivates the terminology introduced in Definition 3.1. For every compact subset $K \subseteq G$ we denote

$$\mathcal{E}'_K(G) := \{u \in \mathcal{E}'(G) \mid \text{supp } u \subseteq K\}.$$

In the case $K = \{\mathbf{1}\}$ we will denote simply $\mathcal{E}'_1(G) := \mathcal{E}'_{\{\mathbf{1}\}}(G)$.

Convolutions. We next wish to introduce the convolution of a smooth function with a distribution with compact support.

Definition 3.5. Let G be any topological group. For all $\varphi \in \mathcal{E}(G)$ define $\check{\varphi} \in \mathcal{E}(G)$ by

$$(\forall x \in G) \quad \check{\varphi}(x) := \varphi(x^{-1}).$$

Then for every $u \in \mathcal{E}'(G)$ we define $\check{u} \in \mathcal{E}'(G)$ by

$$(\forall \varphi \in \mathcal{E}(G)) \quad \check{u}(\varphi) := u(\check{\varphi}).$$

Finally, for all $\varphi \in \mathcal{E}(G)$ and $u \in \mathcal{E}'(G)$ we define their *convolution* as the function

$$\varphi * u: G \rightarrow \mathbb{C}, \quad (\varphi * u)(x) := \check{u}(\varphi \circ L_x)$$

where for all $x \in G$ we define $L_x: G \rightarrow G$, $L_x(y) := xy$.

The above definition is clearly correct, in the sense that $\check{\varphi}, \varphi \circ L_x \in \mathcal{E}(G)$ for all $x \in G$ and $\varphi \in \mathcal{E}(G)$, if G is a Lie group (see also [Eh56]). We will show in Propositions 3.7 and 3.9 below that the definition is actually correct for arbitrary topological groups. To this end we begin by the following simple computation.

Remark 3.6. If $\varphi \in \mathcal{C}^1(G, \mathcal{Y})$, $x \in G$, and $\gamma \in \mathfrak{L}(G)$, then

$$\begin{aligned} (D_\gamma^\lambda \check{\varphi})(x) &= \lim_{t \rightarrow 0} \frac{\varphi(\gamma(-t) \cdot x^{-1}) - \varphi(x^{-1})}{t} \\ &= -\lim_{t \rightarrow 0} \frac{\varphi(\gamma(t) \cdot x^{-1}) - \varphi(x^{-1})}{t} \\ &= -\lim_{t \rightarrow 0} \frac{\varphi(x^{-1} \cdot (\text{Ad}_G(x)\gamma)(t)) - \varphi(x^{-1})}{t} \\ &= -(D_{\text{Ad}_G(x)\gamma}^\lambda \varphi)(x^{-1}). \end{aligned}$$

Similarly, if $n \geq 1$, $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(G)$, then for all $x \in G$ we have

$$(D_{\gamma_1}^\lambda \cdots D_{\gamma_n}^\lambda)(x) = (-1)^n (D_{\text{Ad}_G(x)\gamma_n}^\lambda \cdots D_{\text{Ad}_G(x)\gamma_1}^\lambda \varphi)(x^{-1})$$

(see [BCR81, pag. 45]).

Proposition 3.7. *If G is any topological group, then for all $\varphi \in \mathcal{E}(G)$ we have $\check{\varphi} \in \mathcal{E}(G)$. Moreover, the mapping $\mathcal{E}(G) \rightarrow \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is an isomorphism of locally convex spaces.*

Proof. The linear map $\varphi \mapsto \check{\varphi}$ is equal to its own inverse, hence it suffices to prove that it is continuous. To this end define for arbitrary $n \geq 1$,

$$\begin{aligned}\Psi_n: \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G &\rightarrow \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G, \\ \Psi(\gamma_1, \dots, \gamma_n, x) &= (\text{Ad}_G(x)\gamma_1, \dots, \text{Ad}_G(x)\gamma_n, x^{-1}).\end{aligned}$$

It follows directly by Lemma 2.2 that the mapping Ψ is a homeomorphism. Moreover, by Remark 3.6, it follows that for every $\varphi \in \mathcal{C}^n(G)$ we have

$$(D^\lambda)^n \check{\varphi} = (-1)^n ((D^\lambda)^n \varphi) \circ \Psi_n \quad (3.1)$$

hence $\check{\varphi} \in \mathcal{C}^n(G)$. Since $n \geq 1$ is arbitrary, this shows that if $\varphi \in \mathcal{E}(G)$, then $\check{\varphi} \in \mathcal{E}(G)$.

To check that the linear mapping $\mathcal{E}(G) \rightarrow \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is also continuous, let $k \geq 1$ be any integer and the compact sets $K_1 \subseteq \mathfrak{L}^k(G)$ and $K_2 \subseteq G$ be arbitrary. Define

$$K'_1 := \{(\text{Ad}_G(x)\gamma_1, \dots, \text{Ad}_G(x)\gamma_k) \mid x \in K_1, (\gamma_1, \dots, \gamma_k) \in K_1\}$$

and $K'_2 := \{x^{-1} \mid x \in K_2\}$. Since both the inversion mapping and the adjoint action of G are continuous (Lemma 2.2), it is easily seen that the sets K'_1 and K'_2 are compact. Moreover, it follows by (3.1) along with Definition 3.1 that we have

$$(\forall \varphi \in \mathcal{E}(G)) \quad p_{K_1, K_2}(\check{\varphi}) \leq p_{K'_1, K'_2}(\varphi)$$

hence the linear mapping $\mathcal{E}(G) \rightarrow \mathcal{E}(G)$, $\varphi \mapsto \check{\varphi}$, is indeed continuous. \square

Remark 3.8. If $\varphi \in \mathcal{C}^1(G, \mathcal{Y})$, $x, g \in G$, and $\gamma \in \mathfrak{L}(G)$, then

$$(D_\gamma^\lambda(\varphi \circ L_x))(g) = \lim_{t \rightarrow 0} \frac{\varphi(xg\gamma(t)) - \varphi(xg)}{t} = (D_\gamma^\lambda \varphi)(xg).$$

Therefore $D_\gamma^\lambda(\varphi \circ L_x) = (D_\gamma^\lambda \varphi) \circ L_x$.

Proposition 3.9. *If G is any topological group and \mathcal{Y} is any locally convex space, then for all $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$ and $x \in G$ we have $\varphi \circ L_x \in \mathcal{C}^\infty(G, \mathcal{Y})$.*

Proof. For arbitrary $n \geq 1$ we have the homeomorphism

$$F_n^x: \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G \rightarrow \mathfrak{L}(G) \times \cdots \times \mathfrak{L}(G) \times G, \quad F_n^x(\gamma_1, \dots, \gamma_n, g) = (\gamma_1, \dots, \gamma_n, xg).$$

On the other hand, by iterating Remark 3.8, it follows that for every $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$ we have $(D^\lambda)^n(\varphi \circ L_x) = ((D^\lambda)^n \varphi) \circ F_n^x$ hence $(D^\lambda)^n(\varphi \circ L_x)$ is a continuous function. Since $n \geq 1$ is arbitrary, we obtain $\varphi \circ L_x \in \mathcal{C}^\infty(G, \mathcal{Y})$, and this completes the proof. \square

As already mentioned, the above Propositions 3.7 and 3.9 imply in particular that Definition 3.5 is correct. For later use we now record the version of these results for the multiplication map; see also Remark 5.5 below.

Proposition 3.10. *If G is any topological group with the multiplication $m: G \times G \rightarrow G$, $(x, y) \mapsto xy$, then for any locally convex space \mathcal{Y} the linear mapping*

$$\mathcal{E}(G, \mathcal{Y}) \rightarrow \mathcal{E}(G \times G, \mathcal{Y}), \quad \varphi \mapsto \varphi \circ m$$

is well-defined and continuous.

Proof. Recall from [BCR81, pag. 46] that for every $\varphi \in \mathcal{C}^\infty(G, \mathcal{Y})$, $x, y \in G$, $k \geq 1$, and $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathfrak{L}(G)$ we have

$$\begin{aligned}((D^\lambda)^k(\varphi \circ m))(x, y; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)) \\ = \sum_{\ell=0}^k \sum_{\substack{i_1 < \dots < i_\ell \\ i_{\ell+1} < \dots < i_k}} ((D^\lambda)^\ell \varphi)(xy; \beta_{i_1}, \dots, \beta_{i_\ell}, \text{Ad}_G(y^{-1})\alpha_{i_{\ell+1}}, \dots, \text{Ad}_G(y^{-1})\alpha_{i_k})\end{aligned}$$

where we assume $\{i_1, \dots, i_\ell, i_{\ell+1}, \dots, i_k\} = \{1, \dots, k\}$. With this formula at hand, the continuity of the map $\varphi \mapsto \varphi \circ m$ can be checked just as the continuity of $\varphi \mapsto \tilde{\varphi}$ in the proof of Proposition 3.7. \square

Algebras of local operators.

Definition 3.11. Let G be any topological group. A *local operator* on G is any continuous linear operator $D: \mathcal{E}(G) \rightarrow \mathcal{E}(G)$ with the property

$$(\forall f \in \mathcal{E}(G)) \quad \text{supp}(Df) \subseteq \text{supp } f.$$

We will denote by $\text{Loc}(G)$ the set of all local operators on G . It is easily seen that $\text{Loc}(G)$ is a unital associative algebra of continuous linear operators on the function space $\mathcal{E}(G)$.

Remark 3.12. It follows from [Pee60] that if G is any finite-dimensional Lie group, then $\text{Loc}(G)$ is precisely the set of linear differential operators (possibly of infinite order) on G . Some generalizations of that statement for locally compact groups were obtained in [Ak95, Th. 2.3] and [Ed88, Th. 2.3]. See also [WD73] and [LW11] for some generalizations to the situation when $G = (\mathcal{X}, +)$ for Banach spaces \mathcal{X} that admit suitable bump functions (in particular for Hilbert spaces).

Definition 3.13. Let G be any topological group and recall the notation

$$(\forall x \in G) \quad L_x: G \rightarrow G, \quad L_x(y) = xy.$$

The *left-invariant local operators* on G are the elements of the set

$$\mathcal{U}(G) := \{D \in \text{Loc}(G) \mid (\forall x \in G)(\forall f \in \mathcal{C}^\infty(G)) \quad D(f \circ L_x) = (Df) \circ L_x\}.$$

Note that $\mathcal{U}(G)$ is a unital associative subalgebra of $\text{Loc}(G)$.

For every $\gamma \in \mathfrak{L}(G)$ we have $D_\gamma^\lambda \in \mathcal{U}(G)$ by Remark 3.8. We denote by $\mathcal{U}_0(G)$ the unital associative subalgebra of $\mathcal{U}(G)$ generated by the family $\{D_\gamma^\lambda \mid \gamma \in \mathfrak{L}(G)\}$.

We can now state one of the main problems that have motivated the present paper. We will address this problem in Theorem 5.2 and Corollary 5.3 below.

Problem 3.14. *For any topological group G we have the following inclusions of unital associative algebras:*

$$\mathcal{U}_0(G) \subseteq \mathcal{U}(G) \subseteq \text{Loc}(G).$$

Investigate the gap between $\mathcal{U}_0(G)$ and $\mathcal{U}(G)$, and in particular find necessary or sufficient conditions on G in order to ensure that $\mathcal{U}_0(G) = \mathcal{U}(G)$.

Remark 3.15. If G is any finite-dimensional Lie group, then it follows by the Poincaré-Birkhoff-Witt theorem (see also the above Remark 3.12) that $\mathcal{U}_0(G) = \mathcal{U}(G)$ and this is precisely the complexified universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of the Lie algebra \mathfrak{g} of G . Hence the difficulty of Problem 3.14 lies in the fact that one should extend the Poincaré-Birkhoff-Witt theorem from Lie groups to topological groups.

If G is a pre-Lie group (Definition 5.4) with $\mathfrak{L}(G) = \mathfrak{g}$ or a locally convex Lie group, then [BCR81, Sect. 1.3.1, (4)] shows that the mapping $\mathfrak{L}(G) \rightarrow \mathcal{U}_0(G)$, $\gamma \mapsto D_\gamma^\lambda$ extends to a natural surjective homomorphism of unital associative algebras $U(\mathfrak{g}_\mathbb{C}) \rightarrow \mathcal{U}_0(G)$, whose injectivity can be established under the additional assumption that $\mathcal{C}^\infty(G)$ contains sufficiently many functions, in some sense. (See the method of proof of the Poincaré-Birkhoff-Witt theorem from [CW99] and also [BCR81, Cor. 4.1.1.7].)

For instance, assume that there exists some smooth function on \mathfrak{g} which is equal to 1 on some neighborhood of $0 \in \mathfrak{g}$ and has bounded support. Then the aforementioned natural homomorphism is an isomorphism $U(\mathfrak{g}_\mathbb{C}) \xrightarrow{\sim} \mathcal{U}_0(G)$ for any locally exponential Lie group G (in the sense of [Ne06, Sect. IV]) whose Lie algebra is \mathfrak{g} , which include in particular the

Banach-Lie groups. Note however that this method is not always applicable since there exist Banach spaces that do not admit any nontrivial smooth function with bounded support, for instance $\ell^1(\mathbb{N})$; see [BF66, Sect. 2, Ex. (i)].

On the other hand, a result of the type $U(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} \mathcal{U}_0(G) = \mathcal{U}(G)$ was obtained in [Ak95, Cor. 2.5] in the case when G is any locally compact group, by using the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ discovered in [Glu57] and [La57].

4. EXPONENTIAL LAW FOR SMOOTH FUNCTIONS ON TOPOLOGICAL GROUPS

The main result of this section is Theorem 4.16, which provides a kind of exponential law for smooth functions on topological groups. See for instance [KM97, Ch. I, §3] for a broad discussion on the exponential law for smooth functions on open subsets of locally convex spaces. Further information and references on this topic can be found in [KMR14], [Glö13], and [Al13].

Notation 4.1. Let G and H be arbitrary topological groups. For any locally convex space \mathcal{Y} and any function $\varphi \in \mathcal{C}^\infty(G \times H, \mathcal{Y})$ we define

$$\tilde{\varphi}: G \rightarrow \mathcal{C}^\infty(H, \mathcal{Y}), \quad \tilde{\varphi}(x)(y) = \varphi(x, y).$$

This notation will be preserved throughout the present section.

Some basic formulas on partial derivatives. We now give a definition whose correctness is established in Lemma 4.4 below.

Definition 4.2. Let $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. For $n \geq 1$ we define the partial derivatives $(D_1^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(H) \rightarrow \mathcal{Y}$ and $(D_2^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(G) \rightarrow \mathcal{Y}$ thus:

For $n = 1$, $\beta \in \mathfrak{L}(H)$, $\alpha \in \mathfrak{L}(G)$,

$$\begin{aligned} (D_1^\lambda \varphi)(x, g; \beta) &= \left. \frac{d}{dt} \right|_{t=0} \varphi(x\beta(t), g), \\ (D_2^\lambda \varphi)(x, g; \alpha) &= \left. \frac{d}{dt} \right|_{t=0} \varphi(x, g\alpha(t)). \end{aligned}$$

Furthermore, we define inductively

$$\begin{aligned} ((D_1^\lambda)^{n+1} \varphi)(x, g; \beta_1, \dots, \beta_n, \beta_{n+1}) \\ &= \left. \frac{d}{dt} \right|_{t=0} ((D_1^\lambda)^n \varphi)(x\beta_{n+1}(t), g; \beta_1, \dots, \beta_n) \\ ((D_2^\lambda)^{n+1} \varphi)(x, g; \alpha_1, \dots, \alpha_n, \alpha_{n+1}) \\ &= \left. \frac{d}{dt} \right|_{t=0} ((D_2^\lambda)^n \varphi)(x, g\alpha_{n+1}(t); \alpha_1, \dots, \alpha_n). \end{aligned}$$

Notation 4.3. By $\mathbf{1} \in \mathfrak{L}(G)$ we denote the constant function from \mathbb{R} to G given by $\mathbf{1}(t) = \mathbf{1} \in G$ for all $t \in \mathbb{R}$.

The following lemma ensures the existence and continuity of the maps $(D_1^\lambda)^n \varphi$ and $(D_2^\lambda)^n \varphi$ from Definition 4.2.

Lemma 4.4. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then for all $x \in H$, $g \in G$, $n \geq 1$, $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$, $\alpha_1, \dots, \alpha_n \in \mathfrak{L}(G)$, we have:*

- (a) $((D_1^\lambda)^n \varphi)(x, g; \beta_1, \dots, \beta_n) = ((D^\lambda)^n \varphi)(x, g; (\beta_1, \mathbf{1}), (\beta_2, \mathbf{1}), \dots, (\beta_n, \mathbf{1}))$
- (b) $((D_2^\lambda)^n \varphi)(x, g; \alpha_1, \dots, \alpha_n) = ((D^\lambda)^n \varphi)(x, g; (\mathbf{1}, \alpha_1), (\mathbf{1}, \alpha_2), \dots, (\mathbf{1}, \alpha_n))$
- (c) *The maps $(D_1^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(H) \rightarrow \mathcal{Y}$ and $(D_2^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(G) \rightarrow \mathcal{Y}$ are continuous.*

Proof. Assertions (a) and (b) are straightforward, and (c) follows from (a) and (b), by using the hypothesis $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. \square

Proposition 4.5. *For every $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, $n \geq 1$, and $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$, we have*

$$\begin{aligned} ((D^\lambda)^n \tilde{\varphi})(x; \beta_1, \dots, \beta_n)(g) &= ((D_1^\lambda)^n \varphi)(x, g; \beta_1, \dots, \beta_n) \\ &= ((D^\lambda)^n \varphi)(x, g; (\beta_1, \mathbf{1}), (\beta_2, \mathbf{1}), \dots, (\beta_n, \mathbf{1})). \end{aligned}$$

Proof. The last equality follows from Lemma 4.4(a). For the first equality we will perform the proof by induction on n .

The case $n = 1$: For $x \in H$, $g \in G$ and $\beta \in \mathfrak{L}(H)$ we have

$$(D^\lambda \tilde{\varphi})(x; \beta)(g) = \frac{d}{dt} \Big|_{t=0} \tilde{\varphi}(x\beta(t))(g) = \frac{d}{dt} \Big|_{t=0} \varphi(x\beta(t), g) = (D_1^\lambda \varphi)(x, g; \beta).$$

Now suppose that the assertion was already proved for n . For $n + 1$ we have

$$\begin{aligned} ((D^\lambda)^{n+1} \tilde{\varphi})(x; \beta_1, \dots, \beta_n, \beta_{n+1})(g) &= \frac{d}{dt} \Big|_{t=0} ((D^\lambda)^n \tilde{\varphi})(x\beta_{n+1}(t); \beta_1, \dots, \beta_n)(g) \\ &= \frac{d}{dt} \Big|_{t=0} ((D_1^\lambda)^n \varphi)(x\beta_{n+1}(t), g; \beta_1, \dots, \beta_n) \\ &= ((D_1^\lambda)^{n+1} \varphi)(x, g; \beta_1, \dots, \beta_n, \beta_{n+1}) \end{aligned}$$

and proof ends. \square

Remark 4.6. The formula from Proposition 4.5 gives us the point values of the derivatives of the function $\tilde{\varphi}$ introduced in Notation 4.1. We still have to show that the convergence of the differential quotients to these values holds in the topology of $\mathcal{E}(G, \mathcal{Y})$. This task will be accomplished in Proposition 4.14.

Lemma 4.7. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then for all $x \in H$, $g \in G$, $n \geq 1$, and $\alpha_1, \dots, \alpha_n \in \mathfrak{L}(G)$ we have*

$$\begin{aligned} ((D^\lambda)^n (\tilde{\varphi}(x)))(g; \alpha_1, \dots, \alpha_n) &= ((D_2^\lambda)^n \varphi)(x, g; \alpha_1, \dots, \alpha_n) \\ &= ((D^\lambda)^n \varphi)(x, g; (\mathbf{1}, \alpha_1), (\mathbf{1}, \alpha_2), \dots, (\mathbf{1}, \alpha_n)). \end{aligned}$$

Proof. The proof is similar to the one of Proposition 4.5. \square

Remark 4.8. It follows by Lemma 4.7 and Lemma 4.4(c) that if $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then for all $x \in G$ we have $\tilde{\varphi}(x) := \varphi(x, \cdot) \in \mathcal{C}^\infty(G, \mathcal{Y})$, hence the function $\tilde{\varphi}: H \rightarrow \mathcal{E}(G, \mathcal{Y})$ (see Notation 4.1) is well defined.

Continuity of $\tilde{\varphi}$.

Proposition 4.9. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then the function $\tilde{\varphi}: H \rightarrow \mathcal{E}(G, \mathcal{Y})$ (see Notation 4.1) is continuous.*

Proof. We will show that for any seminorm $p = p_{K_1, K_2}$ on $\mathcal{E}(G, \mathcal{Y})$ as in Definition 3.1 we have $\lim_{x \rightarrow x_0} p(\tilde{\varphi}(x) - \tilde{\varphi}(x_0)) = 0$, which is tantamount to the following condition:

$$(\forall x_0 \in H)(\forall \varepsilon > 0)(\exists U \in \mathcal{V}(x_0))(\forall x \in U) \quad p(\tilde{\varphi}(x) - \tilde{\varphi}(x_0)) \leq \varepsilon.$$

According to the compact sets K_1 and K_2 involved in the definition of the seminorm p , we will analyze separately the two cases that can occur. Let $x_0 \in H$ and $\varepsilon > 0$ be arbitrary, fixed throughout the proof.

Case (a): $p = p_{K_1, K_2}$, where $K_2 \subseteq G$ is any compact set and $K_1 = \emptyset$. If we denote $E(x) := p(\tilde{\varphi}(x) - \tilde{\varphi}(x_0))$, then

$$E(x) = \sup_{g \in K_2} |\tilde{\varphi}(x)(g) - \tilde{\varphi}(x_0)(g)| = \sup_{g \in K_2} |\varphi(x, g) - \varphi(x_0, g)|$$

hence the conclusion follows directly by applying Lemma 2.5 with $x_0 \in X = H$, $T = G$, $K = K_2$ and $f = \varphi: H \times G \rightarrow \mathcal{Y}$, which is a continuous function since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. This completes the proof of Case (a).

Case (b) $p = p_{K_1, K_2}$, for arbitrary compact sets $K_2 \subseteq G$ and $K_1 \subseteq \mathfrak{L}^n(G)$ compact, for some $n \geq 1$.

Denote again $E(x) := p(\tilde{\varphi}(x) - \tilde{\varphi}(x_0))$. In this case we have

$$\begin{aligned} E(x) &= \sup\{|((D^\lambda)^n(\tilde{\varphi}(x) - \tilde{\varphi}(x_0)))(g; \gamma)| \mid g \in K_2, \gamma \in K_1\} \\ &= \sup\{|((D^\lambda)^n(\tilde{\varphi}(x)))(g; \gamma) - ((D^\lambda)^n(\tilde{\varphi}(x_0)))(g; \gamma)| \mid g \in K_2, \gamma \in K_1\}. \end{aligned}$$

By using Lemma 4.7 we obtain

$$E(x) = \sup\{|((D_2^\lambda)^n \varphi)(x, g; \gamma) - ((D_2^\lambda)^n \varphi)(x_0, g; \gamma)| \mid g \in K_2, \gamma \in K_1\}$$

hence the conclusion follows by applying Lemma 2.5 with $x_0 \in X = H$, $T = G \times \mathfrak{L}^n(G)$, and the compact set $K = K_2 \times K_1 \subseteq T$, since $f = (D_2^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(G) \rightarrow \mathcal{Y}$ is a continuous function by Lemma 4.4(c).

This completes the proof. \square

Smoothness of $\tilde{\varphi}$.

Definition 4.10. Let G be any topological group. We say that $\alpha, \beta \in \mathfrak{L}(G)$ *commute* if

$$(\forall s, t \in \mathbb{R}) \quad \alpha(t)\beta(s) = \beta(s)\alpha(t).$$

Remark 4.11. If G and H are any topological groups, then every $\alpha \in \mathfrak{L}(G)$ commutes with $\mathbf{1} \in \mathfrak{L}(G)$, and for every $\alpha \in \mathfrak{L}(G)$ and $\beta \in \mathfrak{L}(H)$ the elements $(\mathbf{1}, \alpha)$ and $(\beta, \mathbf{1})$ from $\mathfrak{L}(H \times G)$ commute.

Lemma 4.12. Let H be any topological group, $n \geq 2$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(H)$. Assume that γ_i commutes with γ_{i+1} for some $i \in \{1, 2, \dots, n-1\}$. Then for any $f \in \mathcal{C}^n(H, \mathcal{Y})$ and $x \in H$ we have

$$\begin{aligned} &((D^\lambda)^n f)(x; \gamma_n, \dots, \gamma_{i+2}, \gamma_{i+1}, \gamma_i, \gamma_{i-1}, \dots, \gamma_1) \\ &= ((D^\lambda)^n f)(x; \gamma_n, \dots, \gamma_{i+2}, \gamma_i, \gamma_{i+1}, \gamma_{i-1}, \dots, \gamma_1). \end{aligned}$$

Proof. The function

$$(t_1, \dots, t_n) \mapsto f(x\gamma_1(t_1)\gamma_2(t_2) \cdots \gamma_n(t_n))$$

belongs to $\mathcal{C}^n(\mathbb{R}^n, \mathcal{Y})$ by [BCR81, Prop. 1.2.2.1]. Therefore we obtain

$$\begin{aligned} &((D^\lambda)^n f)(x; \gamma_n, \dots, \gamma_{i+2}, \gamma_{i+1}, \gamma_i, \gamma_{i-1}, \dots, \gamma_1) \\ &= \frac{\partial^n}{\partial t_1 \cdots \partial t_{i-1} \partial t_i \partial t_{i+1} \partial t_{i+2} \cdots \partial t_n} \Big|_{t_1=\dots=t_n=0} f(x\gamma_1(t_1) \cdots \gamma_n(t_n)) \\ &= \frac{\partial^n}{\partial t_1 \cdots \partial t_{i-1} \partial t_{i+1} \partial t_i \partial t_{i+2} \cdots \partial t_n} \Big|_{t_1=\dots=t_n=0} f(x\gamma_1(t_1) \cdots \gamma_n(t_n)) \\ &= \frac{\partial^n}{\partial t_1 \cdots \partial t_{i-1} \partial t_{i+1} \partial t_i \partial t_{i+2} \cdots \partial t_n} \Big|_{t_1=\dots=t_n=0} \\ &\quad f(x\gamma_1(t_1) \cdots \gamma_{i-1}(t_{i-1})\gamma_{i+1}(t_{i+1})\gamma_i(t_i)\gamma_{i+2}(t_{i+2}) \cdots \gamma_n(t_n)) \\ &= ((D^\lambda)^n f)(x; \gamma_n, \dots, \gamma_{i+2}, \gamma_i, \gamma_{i+1}, \gamma_{i-1}, \dots, \gamma_1) \end{aligned}$$

which is just the required relationship. \square

Lemma 4.13. Let G and H be any topological groups and $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. For some $s \geq 1$ let $\alpha_1, \dots, \alpha_s \in \mathfrak{L}(G)$. Then the following assertions hold for all $x \in H$ and $g \in G$:

(a) For every $\beta \in \mathfrak{L}(H)$ we have

$$((D^\lambda)^{s+1}\varphi)(x, g; (\beta, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) = ((D^\lambda)^{s+1}\varphi)(x, g; (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta, \mathbf{1})).$$

(b) For every $n \geq 1$ and $\beta_1, \dots, \beta_n, \beta_{n+1} \in \mathfrak{L}(H)$ we have

$$\begin{aligned} & ((D^\lambda)^{n+s+1}\varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\beta_{n+1}, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\ &= ((D^\lambda)^{n+s+1}\varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}, \mathbf{1})). \end{aligned}$$

Proof. In both assertions one can start from the right-hand side of the equality to be proved, and one uses Remark 4.11 and Lemma 4.12 for H replaced by $H \times G$ for the pairs $(\mathbf{1}, \alpha_i)$ and $(\beta_{n+1}, \mathbf{1})$. One can thus obtain the order of arguments in the left-hand side of each of the two desired equalities. \square

We are now in a position to solve the problem mentioned in Remark 4.6.

Proposition 4.14. *Let G and H be any topological groups, \mathcal{Y} be any locally convex space and for any $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ define as above $\tilde{\varphi}: H \rightarrow \mathcal{E}(G, \mathcal{Y})$, $\tilde{\varphi}(x)(g) = \varphi(x, g)$. Then for every $x_0 \in G$ and $\beta_1^0, \dots, \beta_{n+1}^0 \in \mathfrak{L}(H)$ we have*

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{((D^\lambda)^n \tilde{\varphi})(x_0 \beta_{n+1}^0(t); \beta_1^0, \dots, \beta_n^0) - ((D^\lambda)^n \tilde{\varphi})(x_0; \beta_1^0, \dots, \beta_n^0)}{t} \\ &= ((D_1^\lambda)^{n+1} \varphi)(x_0, \bullet; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) \end{aligned}$$

in the topology of $\mathcal{E}(G, \mathcal{Y})$ from Definition 3.1.

Proof. Define $h: \mathbb{R} \rightarrow \mathcal{E}(G, \mathcal{Y})$ by

$$h(t) = \begin{cases} \frac{((D^\lambda)^n \tilde{\varphi})(x_0 \beta_{n+1}^0(t); \beta_1^0, \dots, \beta_n^0) - ((D^\lambda)^n \tilde{\varphi})(x_0; \beta_1^0, \dots, \beta_n^0)}{t} & \text{if } t \neq 0, \\ ((D_1^\lambda)^{n+1} \varphi)(x_0, \bullet; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) & \text{if } t = 0. \end{cases}$$

We must prove that $\lim_{t \rightarrow 0} h(t) = h(0)$ in $\mathcal{E}(G, \mathcal{Y})$, that is, for every seminorm $p = p_{K_1, K_2}$ (see Definition 3.1) we have $\lim_{t \rightarrow 0} p(h(t) - h(0)) = 0$.

Depending on the seminorm p , we distinguish two cases that can occur.

Case 1: $p = p_{K_1, K_2}$ for an arbitrary compact set $K_2 \subseteq G$ and $K_1 = \emptyset$.

We denote $E(t) = p(h(t) - h(0))$ and then we have

$$\begin{aligned} E(t) &= \sup\{|h(t)(g) - h(0)(g)| \mid g \in K_2\} \\ &= \sup\left\{\left| \frac{((D_1^\lambda)^n \varphi)(x_0 \beta_{n+1}^0(t), g; \beta_1^0, \dots, \beta_n^0) - ((D_1^\lambda)^n \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0)}{t} \right. \right. \\ &\quad \left. \left. - ((D_1^\lambda)^{n+1} \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) \right| \mid g \in K_2 \right\} \\ &= \sup\{|F(t, g) - F(0, g)| \mid g \in K_2\} \end{aligned}$$

where $F: \mathbb{R} \times G \rightarrow \mathcal{Y}$ is defined by

$$F(t, g) = \begin{cases} \frac{((D_1^\lambda)^n \varphi)(x_0 \beta_{n+1}^0(t), g; \beta_1^0, \dots, \beta_n^0) - ((D_1^\lambda)^n \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0)}{t} & \text{if } t \neq 0, \\ ((D_1^\lambda)^{n+1} \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) & \text{if } t = 0. \end{cases}$$

The desired property $\lim_{t \rightarrow 0} E(t) = 0$ will follow by an application of Lemma 2.5 for $X = \mathbb{R}$, $T = G$, $x_0 = 0 \in \mathbb{R}$, $K = K_2 \subseteq G$ and $f = F: \mathbb{R} \times G \rightarrow \mathcal{Y}$, as soon as we will have checked that F is a continuous function.

To this end, first note that for arbitrary $g \in G$ we have

$$\begin{aligned} \lim_{t \rightarrow 0} F(t, g) &= \frac{d}{dt} \Big|_{t=0} ((D_1^\lambda)^n \varphi)(x_0 \beta_{n+1}^0(t), g; \beta_1^0, \dots, \beta_n^0) \\ &= ((D_1^\lambda)^{n+1} \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) = F(0, g). \end{aligned}$$

Next, we will show that Lemma 2.7 applies for H replaced by $H \times G$, $(x_0, g) \in H \times G$, $X = (\beta_{n+1}^0, \mathbf{1}) \in \mathfrak{L}(H \times G)$, and $f: H \times G \rightarrow \mathcal{Y}$, $f(x, y) = ((D_1^\lambda)^n \varphi)(x, y; \beta_1^0, \dots, \beta_n^0)$, which is continuous since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. Note that the derivative $D_X^\lambda f: H \times G \rightarrow \mathcal{Y}$ is given by $(D_X^\lambda f)(x, y) = ((D_1^\lambda)^{n+1} \varphi)(x, y; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0)$, and this derivative is a continuous function since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$.

Therefore Lemma 2.7 applies and provides a continuous function $\chi: \mathbb{R} \times G \rightarrow \mathcal{Y}$ satisfying for arbitrary $g \in G$ the conditions $\chi(0, g) = 0$ and $f(x_0 \beta_{n+1}^0(t), g) = f(x_0, g) + t(D_X^\lambda f)(x_0, g) + t\chi(t, g)$. We have

$$\begin{aligned} (D_X^\lambda f)(x_0, g) &= (D^\lambda f)(x_0, g; \beta_{n+1}^0, \mathbf{1}) = \frac{d}{dt} \Big|_{t=0} f(x_0 \beta_{n+1}^0(t), g) \\ &= \frac{d}{dt} \Big|_{t=0} ((D_1^\lambda)^n \varphi)(x_0 \beta_{n+1}^0(t), g; \beta_1^0, \dots, \beta_n^0) \\ &= ((D_1^\lambda)^{n+1} \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0) \\ &= F(0, g). \end{aligned}$$

and $F(t, g) = F(0, g) + \chi(t, g)$, hence F is the sum of two continuous functions, since χ is continuous by Lemma 2.7 and $g \mapsto F(0, g) = ((D_1^\lambda)^{n+1} \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0, \beta_{n+1}^0)$ is a continuous function since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$. Therefore F itself is continuous, and this concludes the analysis of Case 1.

Case 2: $p = p_{K_1, K_2}$ for arbitrary compact sets $K_2 \subseteq G$ and $K_1 \subseteq \mathfrak{L}^s(G)$, where $s \geq 1$. Denote again $E(t) = p(h(t) - h(0))$ for $t \in \mathbb{R}$. Then we have

$$E(t) = \sup_{g, \alpha} |((D^\lambda)^s(h(t)))(g; \alpha_1, \dots, \alpha_s) - ((D^\lambda)^s(h(0)))(g; \alpha_1, \dots, \alpha_s)|$$

where $g \in K_2$, $\alpha = (\alpha_1, \dots, \alpha_s) \in K_1$. It then follows that $E(t)$ is the supremum of the values of the seminorm $|\cdot|$ involved in the definition of $p = p_{K_1, K_2}$ (see Definition 3.1) on the vectors in \mathcal{Y} of the form

$$\begin{aligned} &\frac{1}{t} \left(((D^\lambda)^{n+s} \varphi)(x_0 \beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \right. \\ &\quad \left. - ((D^\lambda)^{n+s} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \right) \\ &\quad - ((D^\lambda)^{n+s+1} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\beta_{n+1}^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \end{aligned}$$

where again $g \in K_2$ and $\alpha = (\alpha_1, \dots, \alpha_s) \in K_1$.

Therefore $E(t) = \sup\{|F(t, g, \alpha) - F(0, g, \alpha)| \mid g \in K_2, \alpha \in K_1\}$ where the function $F: \mathbb{R} \times G \times \mathfrak{L}^s(G) \rightarrow \mathcal{Y}$ is given by

$$F(t, g, \alpha) = \begin{cases} \frac{1}{t} \left(((D^\lambda)^{n+s} \varphi)(x_0 \beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \right. \\ \quad \left. - ((D^\lambda)^{n+s} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \right), & \text{if } t \neq 0, \\ ((D^\lambda)^{n+s+1} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\beta_{n+1}^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)), & \text{if } t = 0. \end{cases}$$

The desired property $\lim_{t \rightarrow 0} E(t) = 0$ then follows by an application of Lemma 2.5 for $X = \mathbb{R}$, $T = G \times \mathfrak{L}^s(G)$, $x_0 = 0 \in \mathbb{R}$, the compact $K = K_2 \times K_1 \subseteq G \times \mathfrak{L}^s(G)$ and the function $f = F$, as soon as we will have proved that F is a continuous function.

Just as in Case 1, we first note that

$$\begin{aligned} \lim_{t \rightarrow 0} F(t, g, \alpha) &= \frac{d}{dt} \Big|_{t=0} ((D^\lambda)^{n+s} \varphi)(x_0 \beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\ &= ((D^\lambda)^{n+s+1} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})) \\ &= F(0, g, \alpha) \end{aligned}$$

by using Lemma 4.13(b).

Now let

$$B: \mathbb{R} \rightarrow \mathcal{Y}, \quad B(t) = ((D^\lambda)^{n+s} \varphi)(x_0 \beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)).$$

Since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, we have $B \in \mathcal{C}^1(\mathbb{R}, \mathcal{Y})$, $B'(0) = F(0, g, \alpha)$ (Lemma 4.13(b)) and $B'(t) = ((D^\lambda)^{n+s+1} \varphi)(x_0 \beta_{n+1}^0(t), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1}))$.

We have by the fundamental theorem of calculus for functions with values in the space \mathcal{Y} which may not be complete (see [Gl02a, Th. 1.5])

$$B(t) = B(0) + t \int_0^1 B'(tz) dz = B(0) + t B'(0) + t \int_0^1 B'(tz) dz - t B'(0)$$

and therefore

$$F(t, g, \alpha) = F(0, g, \alpha) + \chi(g, t, \alpha) \tag{4.1}$$

where $\chi: G \times \mathbb{R} \times \mathfrak{L}^s(G) \rightarrow \mathcal{Y}$ is given by

$$\begin{aligned} \chi(g, t, \alpha) &= \int_0^1 ((D^\lambda)^{n+s+1} \varphi)(x_0 \beta_{n+1}^0(tz), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})) dz \\ &\quad - ((D^\lambda)^{n+s+1} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})). \end{aligned}$$

We have $\chi(g, 0, \alpha) = 0$ and χ is continuous by Lemma 2.6 applied for $X = G \times \mathbb{R} \times \mathfrak{L}^s(G)$ and $f: G \times \mathbb{R} \times \mathfrak{L}^s(G) \times [0, 1] \rightarrow \mathcal{Y}$ given by

$$\begin{aligned} f(g, t, \alpha, z) &= ((D^\lambda)^{n+s+1} \varphi)(x_0 \beta_{n+1}^0(tz), g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\beta_{n+1}^0, \mathbf{1})) \end{aligned}$$

which is continuous since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$.

Finally, by using the above equality (4.1), we again obtain that F is the sum of two continuous functions hence is itself continuous, and this completes the proof. \square

Lemma 4.15. *If $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, then the following assertions hold.*

(a) *Let $x \in H$ and $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$ be fixed. Then the function*

$$h := ((D_1^\lambda)^n \varphi)(x, \bullet; \beta_1, \dots, \beta_n): G \rightarrow \mathcal{Y}$$

belongs to $\mathcal{C}^\infty(G, \mathcal{Y})$ and for all $s \geq 1$ and $\alpha_1, \dots, \alpha_s \in \mathfrak{L}(G)$ we have

$$((D^\lambda)^s h)(g; \alpha_1, \dots, \alpha_s) = ((D^\lambda)^{n+s} \varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)).$$

(b) *Let $x \in H$, $\beta_1, \dots, \beta_n \in \mathfrak{L}(H)$, and $\gamma_1, \dots, \gamma_n \in \mathfrak{L}(G)$ be fixed. Then the function*

$$h := ((D^\lambda)^n \varphi)(x, \bullet; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n)): G \rightarrow \mathcal{Y}$$

is in $\mathcal{C}^\infty(G, \mathcal{Y})$ and for every $s \geq 1$ and $\alpha_1, \dots, \alpha_s \in \mathfrak{L}(G)$ we have

$$((D^\lambda)^s h)(g; \alpha_1, \dots, \alpha_s) = ((D^\lambda)^{n+s} \varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)).$$

Proof. Assertion (a) follows by Assertion (b) for $\gamma_1 = \dots = \gamma_n = \mathbf{1} \in \mathfrak{L}(G)$, by using Lemma 4.4(a).

Assertion (b) will be proved by induction on $s \geq 1$. Since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ and $h(g) = ((D^\lambda)^n \varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n))$ it follows that the function h is continuous.

The case $s = 1$: We have

$$\begin{aligned} (D^\lambda h)(g; \alpha) &= \frac{d}{dt} \Big|_{t=0} h(g\alpha(t)) \\ &= \frac{d}{dt} \Big|_{t=0} ((D^\lambda)^n \varphi)(x, g\alpha(t); (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n)) \\ &= ((D^\lambda)^{n+1} \varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha)) \end{aligned}$$

and then since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ we obtain $h \in \mathcal{C}^1(G, \mathcal{Y})$.

Now suppose the assertion was proved for s and we will prove it for $s + 1$. We have

$$\begin{aligned} ((D^\lambda)^{s+1} h)(g; \alpha_1, \dots, \alpha_s, \alpha_{s+1}) &= \frac{d}{dt} \Big|_{t=0} ((D^\lambda)^s h)(g\alpha_{s+1}(t); \alpha_1, \dots, \alpha_s) \\ &= \frac{d}{dt} \Big|_{t=0} ((D^\lambda)^{n+s} \varphi)(x, g\alpha_{s+1}(t); (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s)) \\ &= ((D^\lambda)^{n+s+1} \varphi)(x, g; (\beta_1, \gamma_1), \dots, (\beta_n, \gamma_n), (\mathbf{1}, \alpha_1), \dots, (\mathbf{1}, \alpha_s), (\mathbf{1}, \alpha_{s+1})) \end{aligned}$$

and the proof by induction ends.

Moreover, since $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, we obtain $h \in \mathcal{C}^s(G, \mathcal{Y})$ for every $s \geq 1$. This shows that $h \in \mathcal{C}^\infty(G, \mathcal{Y})$, and the proof is complete. \square

Theorem 4.16. *Let G and H be any topological groups and \mathcal{Y} be any locally convex space. Then for arbitrary $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, the corresponding function*

$$\tilde{\varphi}: H \rightarrow \mathcal{C}^\infty(G, \mathcal{Y}), \quad \tilde{\varphi}(x)(g) := \varphi(x, g).$$

belongs to $\mathcal{C}^\infty(H, \mathcal{E}(G, \mathcal{Y}))$. Moreover, the map

$$\mathcal{E}(H \times G, \mathcal{Y}) \rightarrow \mathcal{E}(H, \mathcal{E}(G, \mathcal{Y})), \quad \varphi \mapsto \tilde{\varphi}$$

is an isomorphism of locally convex spaces.

Proof. To prove the first assertion, let $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$ arbitrary. The fact that $\tilde{\varphi}$ is continuous follows by Proposition 4.9. We will show that for every $n \geq 1$ the derivative $(D^\lambda)^n \tilde{\varphi}: H \times \mathfrak{L}^n(H) \rightarrow \mathcal{E}(G, \mathcal{Y})$ exists and is continuous. The existence of that derivative actually follows from Proposition 4.14. The fact that the derivative takes values in $\mathcal{E}(G, \mathcal{Y})$ is a consequence of Lemma 4.15(a).

For continuity of the above derivative we will prove that for every seminorm $p = p_{K_1, K_2}$ on $\mathcal{E}(G, \mathcal{Y})$ as in Definition 3.1 and every $x_0 \in H$, $\beta_1^0, \dots, \beta_n^0 \in \mathfrak{L}(H)$ and arbitrary $\varepsilon > 0$ there exists a neighborhood U of $(x; \beta_1^0, \dots, \beta_n^0) \in H \times \mathfrak{L}^n(H)$ for which for every $(x; \beta_1, \dots, \beta_n) \in U$ we have $p(((D^\lambda)^n \tilde{\varphi})(x; \beta_1, \dots, \beta_n) - ((D^\lambda)^n \tilde{\varphi})(x_0; \beta_1^0, \dots, \beta_n^0)) \leq \varepsilon$.

Case (a): $p = p_{K_1, K_2}$, where the compact $K_2 \subseteq G$ is arbitrary and $K_1 = \emptyset$.

As in the proof of Proposition 4.9, we denote

$$E(x; \beta_1, \dots, \beta_n) := \sup_{g \in K_2} |((D^\lambda)^n \tilde{\varphi})(x; \beta_1, \dots, \beta_n)(g) - ((D^\lambda)^n \tilde{\varphi})(x_0; \beta_1^0, \dots, \beta_n^0)(g)|.$$

Applying Proposition 4.5 we obtain

$$E(x; \beta_1, \dots, \beta_n) = \sup_{g \in K_2} |((D_1^\lambda)^n \varphi)(x, g; \beta_1, \dots, \beta_n) - ((D_1^\lambda)^n \varphi)(x_0, g; \beta_1^0, \dots, \beta_n^0)|.$$

Now the conclusion follows by applying Lemma 2.5 for $(x_0; \beta_1^0, \dots, \beta_n^0) \in H \times \mathfrak{L}^n(H) = X$, $K = K_2$ compact in $T = G$ and

$$f: H \times \mathfrak{L}^n(H) \times G \rightarrow \mathcal{Y}, \quad f(x; \beta_1, \dots, \beta_n, g) = ((D_1^\lambda)^n \varphi)(x, g; \beta_1, \dots, \beta_n),$$

which is a continuous function since $(D_1^\lambda)^n \varphi: H \times G \times \mathfrak{L}^n(H) \rightarrow \mathcal{Y}$ is continuous by Lemma 4.4(c).

Case (b): $p = p_{K_1, K_2}$ with arbitrary compact sets $K_2 \subseteq G$ and $K_1 \subseteq \mathfrak{L}^s(G)$, where $s \geq 1$.

We denote

$$\begin{aligned} E(x; \beta_1, \dots, \beta_n) = \sup\{ & |((D^\lambda)^s((D^\lambda)^n \tilde{\varphi})(x; \beta_1, \dots, \beta_n))(g; \gamma_1, \dots, \gamma_s) \\ & - ((D^\lambda)^s((D^\lambda)^n \tilde{\varphi})(x_0; \beta_1^0, \dots, \beta_n^0))(g; \gamma_1, \dots, \gamma_s)| \\ & | g \in K_2, \gamma = (\gamma_1, \dots, \gamma_s) \in K_1 \}. \end{aligned}$$

By Proposition 4.5 we obtain

$$\begin{aligned} E(x; \beta_1, \dots, \beta_n) = \sup\{ & |((D^\lambda)^s((D_1^\lambda)^n \varphi)(x, \bullet; \beta_1, \dots, \beta_n))(g; \gamma_1, \dots, \gamma_s) \\ & - ((D^\lambda)^s((D_1^\lambda)^n \varphi)(x_0, \bullet; \beta_1^0, \dots, \beta_n^0))(g; \gamma_1, \dots, \gamma_s)| \\ & | g \in K_2, \gamma = (\gamma_1, \dots, \gamma_s) \in K_1 \}. \end{aligned}$$

Furthermore, by Lemma 4.15(a) we have

$$\begin{aligned} E(x; \beta_1, \dots, \beta_n) = \sup\{ & |((D^\lambda)^{n+s} \varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\mathbf{1}, \gamma_1), \dots, (\mathbf{1}, \gamma_s)) \\ & - ((D^\lambda)^{n+s} \varphi)(x_0, g; (\beta_1^0, \mathbf{1}), \dots, (\beta_n^0, \mathbf{1}), (\mathbf{1}, \gamma_1), \dots, (\mathbf{1}, \gamma_s))| \\ & | g \in K_2, \gamma = (\gamma_1, \dots, \gamma_s) \in K_1 \}. \end{aligned}$$

The conclusion now follows by using Lemma 2.5 for $(x_0; \beta_1^0, \dots, \beta_n^0) \in H \times \mathfrak{L}^n(H) = X$, $T = G \times \mathfrak{L}^s(G)$, $K = K_2 \times K_1$, and $f: H \times \mathfrak{L}^n(H) \times G \times \mathfrak{L}^s(G) \rightarrow \mathcal{Y}$ given by

$$f(x, \beta_1, \dots, \beta_n, g, \gamma_1, \dots, \gamma_s) = ((D^\lambda)^{n+s} \varphi)(x, g; (\beta_1, \mathbf{1}), \dots, (\beta_n, \mathbf{1}), (\mathbf{1}, \gamma_1), \dots, (\mathbf{1}, \gamma_s)).$$

Note that f is continuous since $(D^\lambda)^{n+s} \varphi$ is continuous as a consequence of the hypothesis $\varphi \in \mathcal{C}^\infty(H \times G, \mathcal{Y})$, and this concludes the proof of the fact that $\tilde{\varphi} \in \mathcal{E}(H, \mathcal{E}(G, \mathcal{Y}))$.

For the second assertion, note that the inverse map

$$\mathcal{E}(H, \mathcal{E}(G, \mathcal{Y})) \rightarrow \mathcal{E}(H \times G, \mathcal{Y}), \quad \tilde{\varphi} \mapsto \varphi$$

is well defined, since if $\tilde{\varphi} \in \mathcal{E}(H, \mathcal{E}(G, \mathcal{Y}))$ then $\varphi \in \mathcal{E}(H \times G, \mathcal{Y})$ as an easy consequence of [BCR81, Prop. 1.2.2.3]. Moreover, the continuity of both maps $\varphi \mapsto \tilde{\varphi}$ and $\tilde{\varphi} \mapsto \varphi$ follows easily by taking into account the relations between the derivatives of φ and $\tilde{\varphi}$ provided by Proposition 4.5 and Lemma 4.4 (see also [BCR81, Prop. 1.2.1.5]). This completes the proof. \square

Remark 4.17. It is easily seen that the proof of Theorem 4.16 has a local character, in the sense that it actually leads to a more general result, which can be stated as follows:

Let G and H be any topological groups and \mathcal{Y} be any locally convex space. Pick any open sets $V \subseteq G$ and $W \subseteq H$. Then for arbitrary $\varphi \in \mathcal{C}^\infty(W \times V, \mathcal{Y})$, the corresponding function the function $\tilde{\varphi}: W \rightarrow \mathcal{C}^\infty(V, \mathcal{Y})$, $\tilde{\varphi}(x)(g) := \varphi(x, g)$, belongs to $\mathcal{C}^\infty(W, \mathcal{E}(V, \mathcal{Y}))$. Moreover, the map

$$\mathcal{E}(W \times V, \mathcal{Y}) \rightarrow \mathcal{E}(W, \mathcal{E}(V, \mathcal{Y})), \quad \varphi \mapsto \tilde{\varphi}$$

is an isomorphism of locally convex spaces.

5. STRUCTURE OF INVARIANT LOCAL OPERATORS

In this final section we establish the structure of invariant local operators on any topological group G (Theorem 5.2) and we use that result in order to compare to some extent the two candidates $\mathcal{U}_0(G) \subseteq \mathcal{U}(G)$ to the role of universal enveloping algebra of G ; cf. Problem 3.14. Our main result in this connection is contained in Corollary 5.3 below.

General results.

Proposition 5.1. *If G is any topological group, then for every $f \in \mathcal{E}(G)$ and $u \in \mathcal{E}'(G)$ we have $f * u \in \mathcal{E}(G)$.*

Proof. Let $m: G \times G \rightarrow G$, $m(x, y) = xy$. By denoting $\tilde{u} = v \in \mathcal{E}'(G)$ we have $\tilde{v} = u$ and $(f * u)(x) = \tilde{u}(f \circ L_x) = v(f \circ L_x)$. Now define $\varphi: G \times G \rightarrow \mathbb{C}$, $\varphi(x, y) = f(xy)$. Since $\varphi = f \circ m$, it follows by Proposition 3.10 that $\varphi \in \mathcal{C}^\infty(G \times G)$. If we define $\tilde{\varphi}: G \rightarrow \mathcal{C}^\infty(G)$, $\tilde{\varphi}(x)(y) = \varphi(x, y)$ as in Notation 4.1, then by using Theorem 4.16 we obtain $\tilde{\varphi} \in \mathcal{C}^\infty(G, \mathcal{E}(G))$.

Since $\tilde{\varphi}(x)(y) = f(xy) = (f \circ L_x)(y)$, we have $\tilde{\varphi}(x) = f \circ L_x$, for all $x \in G$, and therefore $f * u = v \circ \tilde{\varphi}$. By using the property $\tilde{\varphi} \in \mathcal{C}^\infty(G, \mathcal{E}(G))$ provided by Theorem 4.16, we obtain $f * u \in \mathcal{E}(G)$, and this completes the proof. \square

We can now prove the following theorem, which extends a well-known property of finite-dimensional Lie groups.

Theorem 5.2. *Let G be any topological group and for every $u \in \mathcal{E}'(G)$ define the linear operator $D_u: \mathcal{C}^\infty(G) \rightarrow \mathcal{C}^\infty(G)$, $D_u f = f * u$*

Then the operator $\Psi: \mathcal{E}'_1(G) \rightarrow \mathcal{U}(G)$, $\Psi(u) = D_u$ is well defined, invertible, and its inverse is

$$\Psi^{-1}: \mathcal{U}(G) \rightarrow \mathcal{E}'_1(G), \quad (\Psi^{-1}(D))(f) = (D\check{f})(\mathbf{1}) \text{ for all } f \in \mathcal{C}^\infty(G) \text{ and } D \in \mathcal{U}(G).$$

Proof. We organize the proof in three steps.

Step 1: We show that Ψ is well defined, that is, for all $u \in \mathcal{E}'_1(G)$ we have $D_u \in \mathcal{U}(G)$. In fact, $D_u(f \circ L_x)(y) = ((f \circ L_x) * u)(y) = \tilde{u}(f \circ L_x \circ L_y)$ and on the other hand $(D_u(f) \circ L_x)(y) = (f * u)(xy) = \tilde{u}(f \circ L_{xy}) = \tilde{u}(f \circ L_x \circ L_y) = D_u(f \circ L_x)(y)$, hence we obtain that $D_u(f \circ L_x) = D_u(f) \circ L_x$.

From $u \in \mathcal{E}'_1(G)$ it follows that $\text{supp } u \subseteq \{\mathbf{1}\} \subseteq G$, hence

$$\text{supp } (D_u f) = \text{supp } (f * u) \subseteq (\text{supp } f)(\text{supp } u)$$

and therefore $\text{supp } (D_u f) \subseteq (\text{supp } f)\{\mathbf{1}\} = \text{supp } f$. We thus obtain that $D_u \in \mathcal{U}(G)$, hence Ψ is well defined.

Step 2: We show that the mapping

$$\Phi: \mathcal{U}(G) \rightarrow \mathcal{E}'_1(G), \quad (\Phi(D))(f) = (D\check{f})(\mathbf{1}) \text{ for all } f \in \mathcal{C}^\infty(G) \text{ and } D \in \mathcal{U}(G)$$

is well defined, that is, for every $D \in \mathcal{U}(G)$ the functional $u: \mathcal{E}(G) \rightarrow \mathbb{C}$, $u(f) = (D\check{f})(\mathbf{1})$, satisfies the condition $u \in \mathcal{E}'_1(G)$.

To this end note that if $\text{supp } f \subseteq U$ then $G \setminus U \subseteq \{x \in G \mid f(x) = 0\}$. Now let $x \in G$ be arbitrary with $x \neq \mathbf{1}$. Since the topology of G is assumed to be separated, there exists some open neighborhood U of x with $\mathbf{1} \notin U$. For every $f \in \mathcal{C}^\infty(G)$ with $\text{supp } f \subseteq U$ we have $\text{supp } \check{f} = (\text{supp } f)^{-1} \subseteq U^{-1}$ and then $\text{supp } (D\check{f}) \subseteq (\text{supp } \check{f}) \subseteq U^{-1}$. We thus obtain $G \setminus U^{-1} \subseteq \{y \in G \mid (D\check{f})(y) = 0\}$.

Since $\mathbf{1} \notin U$ and $\mathbf{1} \notin U^{-1}$, we have $(D\check{f})(\mathbf{1}) = 0$, hence $x \notin \text{supp } u$ for arbitrary $x \in G \setminus \{\mathbf{1}\}$, and then $\text{supp } u \subseteq \{\mathbf{1}\}$. That is, $u \in \mathcal{E}'_1(G)$.

Step 3: We show that $\Psi \circ \Phi = \text{id}_{\mathcal{U}(G)}$ and $\Phi \circ \Psi = \text{id}_{\mathcal{E}'_1(G)}$.

To this end let $D \in \mathcal{U}(G)$ arbitrary and denote $\Phi(D) = u$. We have $u(f) = (D\check{f})(\mathbf{1})$ and $\Psi(u) = D_u$, where

$$(D_u f)(x) = (f * u)(x) = \check{u}(f \circ L_x) = D(f \circ L_x)(\mathbf{1}) = ((Df) \circ L_x)(\mathbf{1}) = (Df)(x).$$

Hence $D_u f = Df$ and $D_u = D$ and we obtain $\Psi \circ \Phi = \text{id}_{\mathcal{U}(G)}$.

Now let $u \in \mathcal{E}'_1(G)$ arbitrary. We have $\Psi(u) = D_u$. Denote $\Phi(D_u) = v \in \mathcal{E}'_1(G)$.

We have $v(f) = (D_u \check{f})(\mathbf{1}) = (\check{f} * u)(\mathbf{1}) = \check{u}(\check{f} \circ L_1) = \check{u}(\check{f}) = u(f)$. Hence $v = u$ and $\Phi \circ \Psi = \text{id}_{\mathcal{E}'_1(G)}$ and the proof is complete. \square

If G is any pre-Lie group, then one can use Theorem 5.2 for endowing $\mathcal{U}(G)$ with a natural topology for which the map Ψ is a homeomorphism if $\mathcal{E}'_1(G)$ carries the weak dual topology which it gets as a closed linear subspace of $\mathcal{E}'(G)$ (see Definition 3.1). That topology of $\mathcal{U}(G)$ can be equivalently described as the locally convex topology determined by the family of seminorms $\{D \mapsto |(Df)(\mathbf{1})|\}_{f \in \mathcal{E}(G)}$.

In the statement of the following corollary, we say that some Banach space \mathcal{X} admits bump functions if there exists $\varphi \in \mathcal{C}^\infty(\mathcal{X})$ which is equal to 1 on some neighborhood of $0 \in \mathcal{X}$, has the support contained in some ball, and for every $k \geq 1$ satisfies the condition $\sup_{x \in \mathcal{X}} \|d_x^k \varphi\| < \infty$. Every Hilbert space admits bump functions; see [WD73] and [LW11] for more details and examples. In this setting, we will provide the following partial answer to Problem 3.14.

Corollary 5.3. *Let G be any Banach-Lie group whose Lie algebra admits bump functions. Then $\mathcal{U}_0(G)$ is a dense subalgebra of $\mathcal{U}(G)$.*

Proof. By the Bahn-Banach theorem, it suffices to check that if $\theta: \mathcal{U}(G) \rightarrow \mathbb{C}$ is any continuous linear functional which vanishes on $\mathcal{U}_0(G)$, then $\theta = 0$. To this end note that, by using the above family of seminorms describing the topology of $\mathcal{U}(G)$, one can find $f \in \mathcal{E}(G)$ with $|\theta(D)| \leq |(Df)(\mathbf{1})|$ for all $D \in \mathcal{U}(G)$. Then the kernel of the linear functional $D \mapsto (Df)(\mathbf{1})$ is contained in $\text{Ker } \theta$ and, since both these kernels are closed 1-codimensional subspaces of $\mathcal{U}(G)$, it follows that, after replacing f by cf for a suitable $c \in \mathbb{C}$, we have $\theta(D) = (Df)(\mathbf{1})$ for all $D \in \mathcal{U}(G)$.

The assumption $\theta(D) = 0$ for all $D \in \mathcal{U}_0(G)$ is then equivalent to the fact that for all $k \geq 1$ we have $(d^k(f \circ \exp_G))(0) = 0$, where $\exp_G: \mathfrak{g} \rightarrow G$ is the exponential map of G , which is a local diffeomorphism at $0 \in \mathfrak{g}$. Now the hypothesis that the Lie algebra \mathfrak{g} admits bump functions allows us to use [LW11, Prop. 3], which ensures that for every local operator T on \mathfrak{g} we have $(T(f \circ \exp_G))(0) = 0$.

We will check that $(Df)(\mathbf{1}) = 0$ for every local operator D on G . To this end pick some open sets U and V for which $\exp_G: V \rightarrow U$ is a diffeomorphism with the inverse denoted by \log_G , where $\mathbf{1} \in U \subseteq G$ and $0 \in V \subseteq \mathfrak{g}$. Then use the hypothesis on \mathfrak{g} to find $\psi \in \mathcal{C}^\infty(\mathfrak{g})$ with $\text{supp } \psi \subseteq V$ and $\psi = 1$ on some neighborhood of $0 \in \mathfrak{g}$. Denote $\phi := \psi \circ \log_G \in \mathcal{C}^\infty(U)$ and extend it with the value 0 on $G \setminus U$. Then $\phi \in \mathcal{C}^\infty(G)$, $\text{supp } \phi \subseteq U$, $\phi = 1$ on some neighborhood of $\mathbf{1} \in U \subseteq G$, and $\psi = \phi \circ \exp_G$. Now define

$$T: \mathcal{C}^\infty(\mathfrak{g}) \rightarrow \mathcal{C}^\infty(\mathfrak{g}), \quad Th = D((\psi h) \circ \log_G) \phi$$

where the function $(\psi h) \circ \log_G \in \mathcal{C}^\infty(V)$ is extended with the value 0 on $\mathfrak{g} \setminus V$. Since D is a local operator, it follows that also T is a local operator, and then by the above observation we obtain $(T(f \circ \exp_G))(0) = 0$, which is equivalent to $(Df)(\mathbf{1}) = 0$. That is, $\theta(D) = 0$ for arbitrary $D \in \mathcal{U}(G)$, and this concludes the proof. \square

Pre-Lie groups. In order to illustrate the above general results and to relate them to the earlier literature, we conclude by some specific examples of topological groups which can be studied from the perspective of the Lie theory (see also Remark 3.15). This is the

case with the class of pre-Lie groups introduced in [BR80] and [BCR81], is closed with respect to several natural operations that may not preserve the locally compact or Lie groups, as for instance taking closed subgroups, infinite direct products, or projective limits ([BCR81, Prop. 1.3.1]). Some specific pre-Lie groups are briefly mentioned in Examples 5.6–5.9 below. See also [HM07] and [Glö02b] for further information on Lie theory for topological groups which may not be Lie groups.

Definition 5.4. A *pre-Lie group* is any topological group G satisfying the conditions:

- (1) The topological space $\mathfrak{L}(G)$ has the structure of a locally convex Lie algebra over \mathbb{R} , whose scalar multiplication, vector addition and bracket satisfy the following conditions for all $t, s \in \mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathfrak{L}(G)$:

$$\begin{aligned} (t \cdot \gamma_1)(s) &= \gamma_1(ts); \\ (\gamma_1 + \gamma_2)(t) &= \lim_{n \rightarrow \infty} (\gamma_1(t/n) \gamma_2(t/n))^n; \\ [\gamma_1, \gamma_2](t^2) &= \lim_{n \rightarrow \infty} (\gamma_1(t/n) \gamma_2(t/n) \gamma_1(-t/n) \gamma_2(-t/n))^{n^2}, \end{aligned}$$

where the convergence is assumed to be uniform on the compact subsets of \mathbb{R} .

- (2) For every nontrivial $\gamma \in \mathfrak{L}(G)$ there exists a function φ of class \mathcal{C}^∞ on some neighborhood of $\mathbf{1} \in G$ such that $(D_\gamma^\lambda \varphi)(\mathbf{1}) \neq 0$.

Remark 5.5. If G is a pre-Lie group, then the multiplication mapping $m: G \times G \rightarrow G$, $(x, y) \mapsto xy$, is smooth by [BCR81, Th. 1.3.2.2 and subsect. 1.1.2] (or alternatively [BR80, Th. and Sect. 1]). In particular, by using the chain rule contained in condition (dcm) of [BCR81, subsect. 1.3.2] (or alternatively the proof of (v) in [BR80, Th.]), we easily recover in this special case the result of Proposition 3.10, to the effect that for any locally convex space \mathcal{Y} the linear mapping $\mathcal{E}(G, \mathcal{Y}) \rightarrow \mathcal{E}(G \times G, \mathcal{Y})$, $\varphi \mapsto \varphi \circ m$, is well-defined and continuous.

Example 5.6. Every locally compact group (in particular, every finite-dimensional Lie group) is a pre-Lie group ([BCR81, pag. 41–41]). In this special case our Theorem 5.2 agrees with [Ed88, Th. 1.4] and [Ak95, Cor. 2.6].

Example 5.7. Every Banach-Lie group is a pre-Lie group ([BCR81, pag. 41–41]). In this special case, we are unable to provide any earlier reference for the result of our Theorem 5.2.

Example 5.8. If \mathcal{X} is any locally convex space, then the abelian locally convex Lie group $(\mathcal{X}, +)$ is a pre-Lie group ([BCR81, pag. 41–41]). In this special case, we are again unable to provide any precise earlier reference for the result provided by our Theorem 5.2. See however [Du73, Th. 3.4] for a related result on real Hilbert spaces.

Example 5.9. Other examples of locally convex Lie groups which are pre-Lie groups are the so-called groups of Γ -rapidly decreasing mappings with values in some Lie groups ([BCR81, Subsect. 4.2.2]).

For the sake of completeness, we will briefly recall the construction of the aforementioned groups of rapidly decreasing mappings, in a very special situation. Let $n \geq 1$ be any integer and $\Gamma = \{\gamma_k \mid k \geq 0\}$, where

$$(\forall k \geq 0) \quad \gamma_k(\cdot) = (1 + |\cdot|)^k$$

and $|\cdot|$ stands for the Euclidean norm on \mathbb{R}^n . Let $(\mathcal{A}, \|\cdot\|)$ be any unital associative Banach algebra with some fixed norm that defines its topology, and \mathcal{A}^\times denote the group of invertible elements in \mathcal{A} , and consider the set of mappings

$$\mathcal{S}(\mathbb{R}^n, \mathcal{A}; \Gamma) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathcal{A}^\times) \mid \lim_{|x| \rightarrow \infty} f(x) = \mathbf{1}; (\forall \alpha \in \mathbb{N}^n) \quad \sup_{\mathbb{R}^n} \gamma_k(\cdot) \|\partial^\alpha f(\cdot)\| < \infty\}$$

endowed with the pointwise multiplication and inversion, where we denote by ∂^α the partial derivatives. Then the group of Γ -rapidly decreasing \mathcal{A}^\times -valued mappings $\mathcal{S}(\mathbb{R}^n, \mathcal{A}; \Gamma)$ has the natural structure of a pre-Lie group. This follows as a very special case of [BCR81, Cor. 4.1.1.7 and Th. 4.2.2.3].

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